

Regularity of harmonic functions for some Markov chains with unbounded range

Fangjun Xu *

Department of Mathematics
University of Kansas
Lawrence, Kansas, 66045 USA

Abstract

We consider a class of continuous time Markov chains on \mathbb{Z}^d . These chains are the discrete space analogue of Markov processes with jumps. Under some conditions, we show that harmonic functions associated with these Markov chains are Hölder continuous.

Keywords: Markov chains, Poincaré inequality, Support theorem, Harmonic functions, Hölder continuity.

Subject Classification: Primary 60J27; Secondary 31B05.

1 Introduction

It is well known that once Harnack inequalities for Markov processes hold, the Hölder regularity of harmonic functions associated with these processes follows. The technique is standard and was first developed by J. Moser in his famous paper [13]. Recent papers [2] and [16] showed that, for some singular Markov processes, the Hölder regularity of harmonic functions still holds while Harnack inequalities fail. To some extent, this means that Harnack inequalities are not necessary needed when proving the Hölder regularity of harmonic functions. So it is natural to ask under what conditions the Hölder regularity of harmonic functions still holds. In this paper, we consider a class of symmetric Markov chains defined from Dirichlet forms and then give conditions for the Hölder regularity of harmonic functions associated with these Markov chains, which are the discrete space analogue of Markov processes with jumps. Our main theorem is, roughly, that an upper bound on the rate of decay of the conductances similar to that of stable processes of index α plus a Poincaré inequality implies that harmonic functions are Hölder continuous. We do not need a lower bound on the rate of decay of the conductances. The main difficulty here is to get near diagonal lower bounds for transition densities. To obtain these lower bounds we use a scaling technique and some weighted Poincaré inequalities. Scaling techniques for Markov chains and Markov processes are widely used when studying heat kernel estimates. For example, [15], [4], [3], [7] and [16]. Weighted Poincaré inequalities are especially helpful when obtaining lower bounds for transition densities. See [9], [14] and references therein.

For each $x \in \mathbb{Z}^d$ and $A \subset \mathbb{Z}^d$, we define $\mu_x = 1$ and $\mu(A) = \sum_{y \in A} \mu_y$. For $x \in \mathbb{Z}^d$ and $r > 0$, let $B(x, r)$ be the open ball in \mathbb{Z}^d centered at x with radius r and $B[x, r]$ the open cube in \mathbb{Z}^d centered

*The author is supported in part by the Robert Adams Fund.

at x with side length $2r$. For each $x, y \in \mathbb{Z}^d$, let $C(x, y)$ be the conductance between x and y . Throughout this paper, we let $\alpha \in (0, 2]$ and assume that the conductance function $C(\cdot, \cdot)$ satisfies the following conditions:

(A1) For any $x, y \in \mathbb{Z}^d$, $C(x, y) = C(y, x) \geq 0$ and $C(x, x) = 0$.

(A2) There exists a positive constant κ_1 such that

$$v_x = \sum_{y \in \mathbb{Z}^d} C(x, y) \geq \kappa_1, \quad \text{for all } x \in \mathbb{Z}^d.$$

(A3) There exist positive constants κ_2 and κ_3 and a nonnegative function $\varphi : \mathbb{N} \rightarrow \mathbb{R}_+$ such that

$$C(x, y) \leq \varphi(|y - x|), \quad \sum_{|z| \geq r} \varphi(|z|) \leq \frac{\kappa_2}{r^\alpha} \quad \text{and} \quad \sum_{|z| < r} |z|^2 \varphi(|z|) \leq \kappa_3 r^{2-\alpha}$$

for all $x, y, z \in \mathbb{Z}^d$ and $r > 0$.

(A4) For any open cube B in \mathbb{Z}^d with side length $2r$, there exist positive constants κ_4 and $\kappa_5 \geq 1$ independent of B such that

$$\sum_B (f(x) - f_B)^2 \leq \kappa_4 r^\alpha \sum_{\kappa_5 B} \sum_{\kappa_5 B} (f(y) - f(x))^2 C(x, y),$$

where $f_B = |B|^{-1} \sum_B f(z)$ with $|B|$ being the cardinality of B , and $\kappa_5 B$ is the cube with the same center as B but side length κ_5 times as large.

Now we use Dirichlet form to define the Markov chain associated with the conductance function $C(\cdot, \cdot)$. For each $f \in L^2(\mathbb{Z}^d, \mu)$, define

$$\begin{cases} \mathcal{E}(f, f) &= \frac{1}{2} \sum_{x, y \in \mathbb{Z}^d} (f(y) - f(x))^2 C(x, y), \\ \mathcal{F} &= \{f \in L^2(\mathbb{Z}^d, \mu) : \mathcal{E}(f, f) < \infty\}. \end{cases}$$

It is easy to see that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{Z}^d, \mu)$. Let X be the continuous time Markov chain corresponding to the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$. In this paper, we consider Markov chains X and show that the harmonic functions associated with X are Hölder continuous under the assumptions (A1)-(A4).

Remark 1.1 *The first two conditions are mild and enable us to define symmetric Markov chains through Dirichlet forms. The last two are used to obtain heat kernel estimates for the Markov chains. In particular, the last condition seems to be necessary for the Hölder regularity of harmonic functions associated with these Markov chains.*

There are some related papers, see [3], [12] and the references therein, in which the regularity of harmonic functions for Markov chains was studied. However, our results are not covered by these works. The differences between this work and [3], [12] are given below.

- (A2) of [3] implies our assumption (A4) with $\alpha = 2$ and $\kappa_5 \geq 1$ through a comparison with the simple random walk. The conductance function C_{xy} in [3] satisfies the above assumptions (A1)-(A4) with $\alpha = 2$. When $\alpha = 2$, our assumption (A3) corresponds to the uniform second

moment condition, which is substantive in [3]. When $0 < \alpha < 2$, our assumption (A3) says that the uniform second moment condition is not needed. Even in the case $\alpha = 2$, our method is a little different from that of [3]. Bass and Kumagai in [3] used the global weighted Poincaré inequality to obtain the near diagonal lower bound while we use local weighted Poincaré inequalities.

- In [12], Hussein and Kassmann considered Markov chains which are similar to stable processes. The essential assumption in [12] is (A3) which concerns the lower bound of the conductances. Our results do not need such assumption. See Example 5.2 for conductances that do not satisfy the assumption (A3) in [12].

The paper is organized as follows. In section 2 we obtain heat kernel estimates for X and then give near diagonal lower bounds for the transition densities of X . In section 3, we prove a support theorem. In section 4, we show the Hölder regularity of harmonic functions associated with X . In section 5, we give a few examples in which assumptions (A1)-(A4) are satisfied.

Throughout this paper, the letter c with or without a subscript indicates a positive constant whose exact value is unimportant and may change from line to line.

2 Heat Kernel Estimates

We start this section with the following Nash inequality.

Proposition 2.1 *There exists c_1 such that*

$$\|f\|_2^{2+\frac{2\alpha}{d}} \leq c_1 \mathcal{E}(f, f) \|f\|_1^{\frac{2\alpha}{d}}, \quad \text{for all } f \in \mathcal{F}.$$

Proof: For any $s > 0$, let $\{Q_i\}_{i=1}^\infty$ be a sequence of open cubes in \mathbb{Z}^d which have equal side length $2s$ and satisfy

$$(1) \ Q_i \cap Q_j = \emptyset \text{ for } i \neq j \quad \text{and} \quad (2) \ \cup_{i=1}^\infty 2Q_i = \mathbb{Z}^d.$$

From assumption (A4),

$$\begin{aligned} \sum_{\mathbb{Z}^d} f^2 &\leq \sum_{i=1}^\infty \sum_{2Q_i} f^2 \\ &\leq 2 \sum_{i=1}^\infty \left(\sum_{2Q_i} (f - f_{2Q_i})^2 + |2Q_i| f_{2Q_i}^2 \right) \\ &\leq 2 \sum_{i=1}^\infty \sum_{2Q_i} (f - f_{2Q_i})^2 + 2 \sum_{i=1}^\infty |2Q_i| f_{2Q_i}^2 \\ &\leq c_2 s^\alpha \sum_{i=1}^\infty \sum_{2\kappa_5 Q_i} \sum_{2\kappa_5 Q_i} (f(x) - f(y))^2 C(x, y) + c_3 s^{-d} \sum_{i=1}^\infty \left(\sum_{2Q_i} |f| \right)^2 \\ &\leq c_4 s^\alpha \mathcal{E}(f, f) + c_5 s^{-d} \|f\|_1^2, \end{aligned}$$

where $f_{2Q_i} = \frac{1}{|2Q_i|} \sum_{2Q_i} |f|$. Therefore, for all $s > 0$, we have

$$\|f\|_2^2 \leq c_4 s^\alpha \mathcal{E}(f, f) + c_5 s^{-d} \|f\|_1^2. \quad (2.1)$$

Choosing s to minimize the right-hand side of (2.1) completes the proof. \square

Write $p(t, x, y)$ for the transition density of X_t .

Proposition 2.2 *There exists c_1 such that*

$$p(t, x, y) \leq c_1(t^{-d/\alpha} \wedge 1), \quad \text{for all } t > 0.$$

Proof: It is obvious that $p(t, x, y) \leq 1$ for all $x, y \in \mathbb{Z}^d$ and $t > 0$. Using Theorem 2.1 in [6] and Proposition 2.1, we know that there exists c such that $p(t, x, y) \leq ct^{-d/\alpha}$ for all $x, y \in \mathbb{Z}^d$ and $t > 0$. Combining these estimates gives the desired result. \square

For each $\rho \geq 1$, set $\mathcal{S} = \rho^{-1}\mathbb{Z}^d$. For each $x \in \mathcal{S}$ and $A \subset \mathcal{S}$, let $\mu_x^\rho = \rho^{-d}$ and $\mu^\rho(A) = \sum_{y \in A} \mu_y^\rho$. Define the rescaled process V as

$$V_t = \rho^{-1}X_{\rho^\alpha t}, \quad t \geq 0.$$

Using similar arguments as in [16], we see that the Dirichlet form corresponding to V is

$$\begin{cases} \mathcal{E}^\rho(f, f) &= \sum_{\mathcal{S}} \sum_{\mathcal{S}} (f(y) - f(x))^2 C^\rho(x, y), \\ \mathcal{F}_\rho &= \{f \in L^2(\mathcal{S}, \mu^\rho) : \mathcal{E}^\rho(f, f) < \infty\}, \end{cases}$$

where $C^\rho(x, y) = \rho^{\alpha-d}C(\rho x, \rho y)$ for all $x, y \in \mathcal{S}$.

Write $p^\rho(t, \cdot, \cdot)$ for the transition density of V_t . Then we have

$$p^\rho(t, x, y) = \rho^d p(\rho^\alpha t, \rho x, \rho y) \quad (2.2)$$

for all $x, y \in \mathcal{S}$ and $t \geq 0$. The process V satisfies the following Poincaré inequality.

Lemma 2.3 *For any open cube B in \mathcal{S} with side length $2r$, there is a constant c independent of B and ρ such that*

$$\sum_B (f(x) - f_B)^2 \rho^{-d} \leq cr^\alpha \sum_{\kappa_5 B} \sum_{\kappa_5 B} (f(y) - f(x))^2 C^\rho(x, y).$$

Proof: This follows from assumption (A4) and change of variables. \square

For λ large enough, let V^λ be the process V with jumps larger than λ removed. Write $p^{\rho, \lambda}(t, x, y)$ for the transition density of V_t^λ .

Proposition 2.4 *There exists c_1 independent of ρ and λ such that*

$$p^{\rho, \lambda}(t, x, y) \leq c_1 t^{-d/\alpha} e^t.$$

Proof: Under the first two parts of assumption (A3), the above upper bound follows easily from Theorem 2.1 in [6], Lemma 2.3 and the proof of Proposition 2.1. \square

We can obtain a better upper bound for the transition density of V^λ .

Lemma 2.5 *There exist c_1 and c_2 independent of ρ and λ such that*

$$p^{\rho,\lambda}(t, x, y) \leq c_1 t^{-\frac{d}{\alpha}} e^{c_2 t} e^{-|x-y|/\lambda}.$$

Proof: Applying Theorem 3.25 in [6] and Proposition 2.4, we have

$$p^{\rho,\lambda}(t, x, y) \leq c_2 t^{-d/\alpha} e^{t-E(2t,x,y)}, \quad (2.3)$$

where

$$\begin{aligned} E(t, x, y) &= \sup \{ |\psi(x) - \psi(y)| - t\Lambda(\psi)^2 : \Lambda(\psi) < \infty \}, \\ \Lambda(\psi)^2 &= \|e^{-2\psi} \Gamma_\lambda[e^\psi]\|_\infty \vee \|e^{2\psi} \Gamma_\lambda[e^{-\psi}]\|_\infty, \\ \Gamma_\lambda[v](\xi) &= \sum_{\eta \in \mathcal{S}, |\eta-\xi| \leq \lambda} (v(\eta) - v(\xi))^2 C(\rho\xi, \rho\eta) \rho^\alpha. \end{aligned}$$

Let $\psi(\xi) = \lambda^{-1}(|\xi - x| \wedge |y - x|)$. Then $|\psi(\eta) - \psi(\xi)| \leq |\eta - \xi|/\lambda$ and

$$(e^{\psi(\eta)-\psi(\xi)} - 1)^2 \leq |\psi(\eta) - \psi(\xi)|^2 e^{2|\psi(\eta)-\psi(\xi)|} \leq c_3 |\eta - \xi|^2 / \lambda^2$$

for all $\eta, \xi \in \mathcal{S}$ with $|\eta - \xi| \leq \lambda$. Therefore

$$e^{-2\psi(\xi)} \Gamma_\lambda[e^\psi](\xi) = \sum_{\eta \in \mathcal{S}, |\xi-\eta| \leq \lambda} (e^{\psi(\eta)-\psi(\xi)} - 1)^2 C(\rho\eta, \rho\xi) \rho^\alpha \leq c_4 \lambda^{-\alpha} \leq c_4.$$

In the last second inequality we used the last part of assumption (A3). The same upper bound is obtained if ψ is replaced by $-\psi$. Note that $|\psi(x) - \psi(y)| = |x - y|/\lambda$. Substituting these estimates into (2.3), we have our result after doing some algebra. \square

For any set $A \subset \mathbb{Z}^d$, let

$$T_A = \inf \{t \geq 0 : X_t \notin A\} \quad \text{and} \quad \tau_A = \inf \{t \geq 0 : X_t \in A\}.$$

The upper bound in Lemma 2.5 implies the following key exit time estimates for X . The proof is the same as the one given in Proposition 3.4 of [3] except some minor modifications.

Theorem 2.6 *For $a > 0$ and $0 < b < 1$, there exists $\gamma = \gamma(a, b) \in (0, 1)$ such that for every $R > 0$ and $x \in \mathbb{Z}^d$,*

$$\mathbb{P}^x(\tau_{B(x, aR)}(X) < \gamma R^\alpha) \leq b.$$

Next we are going to obtain near diagonal lower bounds for the transition densities of X .

Proposition 2.7 *The following two statements are equivalent:*

(1) *There is an ϵ such that*

$$p(t, x, y) \geq \epsilon t^{-d/\alpha}$$

for all $t \geq 1$ and $|y - x| \leq 2t^{1/\alpha}$.

(2) *There is an ϵ such that*

$$p^\rho(1, \rho^{-1}x, \rho^{-1}y) \geq \epsilon$$

for all $\rho \geq 1$ and $|\rho^{-1}y - \rho^{-1}x| \leq 2$.

Proof: This follows easily from (2.2) and change of variables. \square

Remark 2.8 *In fact, statements (1) and (2) are also equivalent to the following one: There is an ϵ such that*

$$p^\rho(t, \rho^{-1}x, \rho^{-1}y) \geq \epsilon t^{-d/\alpha}$$

for all $t \geq \rho^{-\alpha}$, $\rho \geq 1$ and $|\rho^{-1}y - \rho^{-1}x| \leq 2t^{1/\alpha}$.

In the remainder of this section, we first prove the statement (2) in Proposition 2.7 and then obtain the near diagonal lower bound for the transition densities of X .

For any $R > 0$ and $x_0 \in \mathcal{S}$, let $B = B[x_0, R]$ be the open cube in \mathcal{S} centered at x_0 with side length $2R$,

$$\phi_R(x) = c_1 \left(R^2 - |x_0 - x|_m^2 \right)^+ \quad (2.4)$$

where $|x_0 - x|_m = \max\{|x_0^1 - x^1|, \dots, |x_0^d - x^d|\}$ and c_1 is chosen so that $\sum_B \phi_R(x) = \rho^d$, and set

$$\bar{f} = \sum_B f(x) \phi_R(x) \rho^{-d}.$$

Then we have the following local weighted Poincaré inequality with its proof given in Appendix Two.

Proposition 2.9 *For any $\rho \geq 1$, there exists a constant c_1 independent of ρ and R such that*

$$\sum_B (f(x) - \bar{f})^2 \phi_R(x) \rho^{-d} \leq c_1 R^\alpha \sum_B \sum_B (f(x) - f(y))^2 (\phi_R(x) \wedge \phi_R(y)) C^\rho(x, y)$$

for all $R \in \bigcup_{n=0}^{\infty} [\frac{n}{\rho} + \frac{1}{4\rho}, \frac{n}{\rho} + \frac{1}{\rho}]$.

We now consider V killed on exiting B . Since

$$\mathbb{P}^x(V_t \in A, \tau_B > t) \leq \mathbb{P}^x(V_t \in A) = \sum_A p^\rho(t, x, y) \mu_y^\rho,$$

this means that $\mathbb{P}^x(V_t = y, \tau_B > t)$ has a density bounded by $p^\rho(t, x, y)$. Write $p_B^\rho(t, x, y)$ for the density of $\mathbb{P}^x(V_t = y, \tau_B > t)$. Then we can use Proposition 2.9 to get lower bound for the transition density $p_B^\rho(1, x, y)$ when x and y are not far away. See the following proposition for details. The proof of the following proposition is long and similar to that of Proposition 4.9 in [1], Theorem 3.4 in [8], and Theorem 2.5 in [10]. We postpone it to Appendix One.

Proposition 2.10 *For $R \in [2d, 4d]$, there exists c_1 independent of ρ , x_0 and R such that*

$$p_B^\rho(1, x, y) \geq c_1,$$

for every $(x, y) \in B(x_0, 3R/4) \times B(x_0, 3R/4)$.

Theorem 2.11 *There is an ϵ such that*

$$p(t, x, y) \geq \epsilon t^{-d/\alpha}$$

for all $t \geq 1$ and $|y - x| \leq 2t^{1/\alpha}$.

Proof: From the argument before Proposition 2.10, we see that $p^\rho(1, x, y) \geq p_B^\rho(1, x, y)$ for all $x, y \in \mathcal{S}$. Then using Propositions 2.7 and 2.10 gives the desired near diagonal lower bound for $p(t, x, y)$. \square

3 Support Theorem

Lemma 3.1 *Given $\delta > 0$, there exists κ such that if $x, y \in \mathbb{Z}^d$ and $A \subset \mathbb{Z}^d$ with $\text{dist}(x, A)$ and $\text{dist}(y, A)$ both larger than $\kappa t^{1/\alpha}$, then*

$$\mathbb{P}^x(X_t = y, T_A \leq t) \leq \delta t^{-d/\alpha}. \quad (3.1)$$

Proof: Let $S_A = \sup\{s \leq t : X_s \in A\}$ be the last hitting time of A before time t . Then

$$\begin{aligned} \mathbb{P}^x(X_t = y, t/2 \leq T_A \leq t) &\leq \mathbb{P}^x(X_t = y, t/2 \leq S_A \leq t) \\ &= \mathbb{P}^y(X_t = x, T_A \leq t/2). \end{aligned}$$

The last equation follows from time reversal, see Lemma 4.5 of [3]. Using strong Markov property and Proposition 2.2, we have

$$\begin{aligned} \mathbb{P}^y(X_t = x, T_A \leq t/2) &= \mathbb{P}^y(1_{\{T_A \leq t/2\}} \mathbb{P}^{X_{T_A}}(X_{t-T_A} = x)) \\ &\leq c_1(t/2)^{-d/\alpha} \mathbb{P}^y(T_A \leq t/2) \\ &\leq c_1(t/2)^{-d/\alpha} \mathbb{P}^y(\tau_{B(y, \kappa t^{1/\alpha})} \leq t/2) \\ &\leq \delta t^{-d/\alpha}. \end{aligned}$$

Here we used Theorem 2.6 in the last inequality by choosing proper κ . Similarly,

$$\mathbb{P}^x(X_t = y, T_A \leq t/2) \leq \delta t^{-d/\alpha}.$$

Combining these estimates gives our result. \square

Proposition 3.2 *For all $t \geq 1$, there exist c_1 and $\theta \in (0, 1)$ such that if $|x - z|, |y - z| \leq t^{1/\alpha}$, $x, y, z \in \mathbb{Z}^d$ and $r \geq t^{1/\alpha}/\theta$, then*

$$\mathbb{P}^x(X_t = y, \tau_{B(z, r)} > t) \geq c_1 t^{-d/\alpha}. \quad (3.2)$$

Proof: Choose $\delta = \epsilon/2$ in Lemma 3.1. Then for $r > (\kappa + 1)t^{1/\alpha}$ we have

$$\begin{aligned} &\mathbb{P}^x(X_t = y, \tau_{B(z, r)} > t) \\ &= \mathbb{P}^x(X_t = y) - \mathbb{P}^x(X_t = y, \tau_{B(z, r)} \leq t) \\ &\geq \frac{\epsilon}{2} t^{-d/\alpha}. \end{aligned}$$

Here we used Theorem 2.11 in the last inequality. \square

Remark 3.3 *The above proposition still holds if we replace “ $|x - z|, |y - z| \leq t^{1/\alpha}$, $x, y, z \in \mathbb{Z}^d$ ” with “ $|x - y| \leq 2t^{1/\alpha}$, $x, y \in \mathbb{Z}^d$ ” and “ z ” in (3.2) with “ x ”, respectively.*

Corollary 3.4 *For each $\epsilon \in (0, 1)$, there exists $\theta = \theta(\epsilon) \in (0, 1)$ with the following property: if $x, y \in \mathbb{Z}^d$ with $|x - y| < t^{1/\alpha}$, $t \in [1, \theta^\alpha r^\alpha]$, and $\Gamma \subset B(y, t^{1/\alpha})$ satisfies $\mu(\Gamma)t^{-d/\alpha} \geq \epsilon$, then*

$$\mathbb{P}^x(X_t \in \Gamma \text{ and } \tau_{B(x, r)} > t) \geq c_1 \epsilon. \quad (3.3)$$

Proof: This follows easily from Proposition 3.2 and Remark 3.3. \square

Remark 3.5 *In fact, the condition “ $t \in [1, \theta^\alpha r^\alpha)$ ” in the above corollary can be relaxed to “ $t \in [0, \theta^\alpha r^\alpha)$ ”.*

Proposition 3.6 *For each $\epsilon \in (0, 1)$, there exist constants c_1 and $\eta = \eta(\epsilon) \in (0, 1)$ such that for any $x \in \mathbb{Z}^d$, if $A \subset B(x, \eta r)$ satisfies $\mu(A)/\mu(B(x, \eta r)) \geq \epsilon$, then*

$$\mathbb{P}^x(T_A < \tau_{B(x, r)}) \geq c_1 \epsilon. \quad (3.4)$$

Proof: Choose $\eta = 2^{-\alpha} \theta$ and $t = (\eta r)^\alpha$. The above proposition follows from Corollary 3.4 and Remark 3.5. \square

4 Hölder Continuity

The following lemma can be easily proved by using Propositions 2.2 and 3.2. We refer to Lemma 5.2 in [3] for its proof.

Lemma 4.1 *There exist constants c_1 and c_2 such that*

$$c_1 r^\alpha \leq \mathbb{E}^x \tau_{B(x, r)} \leq c_2 r^\alpha.$$

Since X is a Hunt process, there is a Lévy system formula for it. We refer to [7] for its proof.

Lemma 4.2 *For any nonnegative function f on $\mathbb{Z}^d \times \mathbb{Z}^d$ that vanishes on the diagonal and any stopping time T ,*

$$\mathbb{E}^x \left[\sum_{s \leq T} f(X_{s-}, X_s) \right] = \mathbb{E}^x \left[\int_0^T \sum_{y \in \mathbb{Z}^d} f(X_s, y) C(x, y) ds \right].$$

We say that h is harmonic with respect to X in a domain D if $h(X_{t \wedge \tau_D})$ is a \mathbb{P}^x -martingale for every x in D .

Theorem 4.3 *Suppose that h is bounded on \mathbb{Z}^d and harmonic in $B(x_0, r)$ with respect to the process X . Then there exist constants c and $\beta \in (0, \alpha)$ such that*

$$|h(x) - h(y)| \leq c \left(\frac{|x - y|}{r} \right)^\beta \sup |h|.$$

Proof: Without loss of generality, we assume that $0 \leq h \leq 1$ on \mathbb{Z}^d . From Proposition 3.6 we know that there exist constants c_1 and η such that if $A \subset B(x, \eta r)$ with $|A|/|B(x, \eta r)| \geq 1/4$, then

$$\mathbb{P}^x(T_A < \tau_{B(x, r)}) \geq c_1.$$

From Lemmas 4.1 and 4.2, there exists c_2 such that

$$\mathbb{P}^x(X_{\tau_{B(x, r)}} \notin B(x, s)) \leq c_2 \left(\frac{r}{s} \right)^\alpha, \quad \text{for all } s \geq 2r.$$

Let $\gamma = 1 - \frac{c_1}{4}$ and $\rho = \eta \wedge \left(\frac{\gamma}{2}\right)^{1/\alpha} \wedge \left(\frac{c_1\gamma^2}{8c_2}\right)^{1/\alpha}$. We need to show

$$\sup_{B(x, \rho^k r)} h - \inf_{B(x, \rho^k r)} h \leq \gamma^k, \quad \text{for all } k.$$

For simplicity of notation, set

$$B_i = B(x, \rho^i r), \quad \tau_i = \tau_{B_i}, \quad a_i = \sup_{B_i} h, \quad \text{and} \quad b_i = \inf_{B_i} h.$$

By the assumption that $0 < h < 1$ on \mathbb{Z}^d , we see $a_i - b_i \leq 1 \leq \gamma^i$ for $i \leq 0$. Suppose $a_i - b_i \leq \gamma^i$ for $i \leq k$. Now we only need to prove

$$a_{k+1} - b_{k+1} \leq \gamma^{k+1}.$$

Notice that $b_k \leq h \leq a_k$ on B_{k+1} . Define

$$A = \left\{ z \in B(x, \rho^{k+1} r) : h(z) \leq \frac{a_k + b_k}{2} \right\}.$$

We can assume $\mu(A)/\mu(B(x, \rho^{k+1} r)) \geq 1/2$. Otherwise we use $1 - h$ instead of h in the above definition of A . By the definition of a_{k+1} and b_{k+1} , we can choose $z_1, z_2 \in B_{k+1}$ such that $a_{k+1} = h(z_1)$ and $b_{k+1} = h(z_2)$. By optional stopping,

$$\begin{aligned} h(z_1) - h(z_2) &= \mathbb{E}^{z_1} [h(X_{T_A \wedge \tau_k}) - h(z_2)] \\ &= \mathbb{E}^{z_1} [h(X_{T_A}) - h(z_2); T_A < \tau_k] \\ &\quad + \mathbb{E}^{z_1} [h(X_{\tau_k}) - h(z_2); T_A > \tau_k, X_{\tau_k} \in B_{k-1}] \\ &\quad + \sum_{i=1}^{\infty} \mathbb{E}^{z_1} [h(X_{\tau_k}) - h(z_2); T_A > \tau_k, X_{\tau_k} \in B_{k-1-i} - B_{k-i}] \\ &\leq \left(\frac{a_k + b_k}{2} - b_k \right) \mathbb{E}^{z_1} (T_A < \tau_k) + (a_k - b_k) \mathbb{P}^{z_1} (T_A > \tau_k) \\ &\quad + \sum_{i=1}^{\infty} (a_{k-1-i} - b_{k-1-i}) \mathbb{P}^{z_1} (X_{\tau_k} \notin B_{k-i}) \\ &\leq (a_k - b_k) \left(1 - \frac{\mathbb{P}^{z_1} (T_A < \tau_k)}{2} \right) + \sum_{i=1}^{\infty} c_2 \gamma^{k-1} (\rho^\alpha / \gamma)^i \\ &\leq \left(1 - \frac{c_1}{2} \right) \gamma^k + 2c_2 \gamma^{k-2} \rho^\alpha \\ &\leq \left(1 - \frac{c_1}{2} \right) \gamma^k + \frac{c_1}{4} \gamma^k \\ &= \gamma^{k+1}. \end{aligned}$$

For any $x, y \in B(x_0, r)$, let k be the smallest integer such that $|y - x| < \rho^k r$. Then $\log(|x - y|) \geq (k + 1) \log \rho + \log r$ and

$$|h(y) - h(x)| \leq e^{k \log \gamma} \leq c_3 e^{\left(\frac{\log \gamma}{\log \rho}\right) \log \left(\frac{|x-y|}{r}\right)} = c_3 \left(\frac{|x-y|}{r}\right)^{\frac{\log \gamma}{\log \rho}}.$$

By the definition of γ and ρ , it is easy to see that $\log \gamma / \log \rho \in (0, \alpha)$. Our result follows with $\beta = \log \gamma / \log \rho$. \square

5 Examples

In this section, we give conductance functions which satisfy assumptions (A1)-(A4).

Example 5.1 For $\alpha \in (0, 2)$ and $d \geq 2$, we define the conductance functions $C_{\alpha,1}(\cdot, \cdot)$ by

$$C_{\alpha,1}(x, y) = \begin{cases} \frac{c(x, y)}{|x - y|^{d + \alpha}} & \text{if } y \neq x; \\ 0 & \text{otherwise,} \end{cases}$$

where $c(x, y) = c(y, x)$ and $0 < c_1 \leq c(x, y) \leq c_2 < \infty$ for all $x, y \in \mathbb{Z}^d$. The parabolic Harnack inequality holds for the Markov chains corresponding to $C_{\alpha,1}(\cdot, \cdot)$ (see, [5]).

Example 5.2 For $\alpha \in (0, 2)$ and $d \geq 2$, let \mathbb{Z}_i be the i -th coordinate axis in \mathbb{Z}^d . We define the conductance functions $C_{\alpha,2}(\cdot, \cdot)$ by

$$C_{\alpha,2}(x, y) = \begin{cases} \frac{c(x, y)}{|x - y|^{1 + \alpha}} & \text{if } y - x \in \bigcup_{i=1}^d \mathbb{Z}_i \setminus \{0\}; \\ 0 & \text{otherwise,} \end{cases}$$

where $c(x, y) = c(y, x)$ and $0 < c_1 \leq c(x, y) \leq c_2 < \infty$ for all $x, y \in \mathbb{Z}^d$. The Markov chains corresponding to $C_{\alpha,2}(\cdot, \cdot)$ are the discrete space analogue of the singular stable-like processes in [2] and [16]. When $c(x, y) \equiv 1$, the Markov chain corresponding to $C_{\alpha,2}(\cdot, \cdot)$ is the discrete space analogue of the d -dimensional Lévy process whose coordinate processes are independent 1-dimensional symmetric α -stable processes.

Example 5.3 For $d \geq 3$, let e^i be the unit vector in \mathbb{R}^d with the i -th coordinate being 1. Let $b_n = n^{n_n}$ and a_n be two sequences of positive numbers with $\sum_{n=1}^{\infty} a_n \leq 1/8$ and $\sum_{n=1}^{\infty} a_n b_n^2 < \infty$. Let $\epsilon = 2 \sum_{n=1}^{\infty} a_n$. We define the conductance function $C_{2,3}(\cdot, \cdot)$ by

$$C_{2,3}(x, y) = \begin{cases} a_n & \text{if } y - x = \pm b_n e^1; \\ \frac{1 - \epsilon}{2(d-1)} & \text{if } y - x = \pm e^j \text{ and } j = 2, \dots, d; \\ 0 & \text{otherwise.} \end{cases}$$

This example is from [3]. The conductance function $C_{2,3}(\cdot, \cdot)$ satisfies assumptions (A1)-(A4) with $\alpha = 2$. The uniform Harnack inequality does not hold for the Markov chain corresponding to $C_{2,3}(\cdot, \cdot)$ (see, [3]).

6 Appendix One

The goal of this section is to prove Proposition 2.10. Recall the definition of the transition density $p_B^\rho(t, x, y)$ of V killed upon exiting the open cube B in \mathcal{S} with center x_0 and side length $2R \in [4d, 8d]$. Notice that the Dirichlet form for V^B (V killed upon exiting the open cube B) is $(\mathcal{E}^\rho, \mathcal{F}_\rho^B)$ where

$$\mathcal{F}_\rho^B = \{f : \mathcal{F}_\rho : f = 0 \text{ on } B^c\}.$$

So for $f \in \mathcal{F}_\rho^B$,

$$\mathcal{E}^\rho(f, f) = \sum_B \sum_B (f(x) - f(y))^2 C^\rho(x, y) + \sum_B f(x)^2 \kappa_B(x) \mu_x^\rho,$$

where $\kappa_B(x) = 2 \sum_{B^c} C(\rho x, \rho y) \mu_y^\rho$.

Lemma 6.1 *There exists a positive constant c_1 independent of ρ and B such that*

$$p_B^\rho(t, x, y) \leq c_1 t^{-d/\alpha} \quad \text{and} \quad \left| \frac{\partial p_B^\rho(t, x, y)}{\partial t} \right| \leq c_1 t^{-1-\frac{d}{\alpha}}$$

for all $x, y \in B$ and $t > 0$.

Proof: The first inequality follows immediately from Proposition 2.2 and the argument before Proposition 2.10. Since

$$\sum_B \sum_B p_B^\rho(t, x, y)^2 \mu_x^\rho \mu_y^\rho \leq \sum_B p_B^\rho(2t, x, x)^2 \mu_x^\rho < \infty,$$

the symmetric semigroup P_t^B of V^B is a Hilbert-Schmidt operator on $L^2(B, \mu^\rho)$ and so it is compact and has a discrete spectrum $\{e^{-\lambda_i t}, 1 \leq i \leq N\}$, with repetitions according to multiplicity. Here N is a natural number determined by the Hilbert space $L^2(B, \mu^\rho)$. Let $\{\phi_i, 1 \leq i \leq N\}$ be the corresponding eigenfunctions normalized to have unit L^2 -norm on B and to be orthogonal to each other. Then

$$p_B^\rho(t, x, y) = \sum_{i=1}^N e^{-\lambda_i t} \phi_i(x) \phi_i(y).$$

Hence $p_B^\rho(t, x, y)$ is differential with respect to t and

$$\begin{aligned} \left| \frac{\partial p_B^\rho(t, x, y)}{\partial t} \right| &= \left| - \sum_{i=1}^N \lambda_i e^{-\lambda_i t} \phi_i(x) \phi_i(y) \right| \\ &\leq \sum_{i=1}^N \lambda_i e^{-\lambda_i t/2} e^{-\lambda_i t/2} |\phi_i(x)| |\phi_i(y)| \\ &\leq \frac{c_2}{t} \left(\sum_{i=1}^N e^{-\lambda_i t/2} \phi_i(x)^2 \right)^{1/2} \left(\sum_{i=1}^N e^{-\lambda_i t/2} \phi_i(y)^2 \right)^{1/2} \\ &= \frac{c_2}{t} \left(p_B^\rho(t/2, x, x) \right)^{1/2} \left(p_B^\rho(t/2, y, y) \right)^{1/2} \\ &\leq c_3 t^{-1-\frac{d}{\alpha}}. \end{aligned}$$

Here we used the fact that $h(x) = x e^{-xt/2}$ is bounded on $[0, \infty)$ by c_2/t .

For $\epsilon \in (0, 1)$, define

$$G(t) = \sum \phi_R(x) \log p_B^{\rho, \epsilon}(t, x, y_0) \mu_x^\rho,$$

where $p_B^{\rho, \epsilon}(t, x, y) = p_B^\rho(t, x, y) + \epsilon$.

Lemma 6.2 *Fix $y_0 \in B$. Then, for every $t > 0$,*

$$G'(t) = -\mathcal{E}^\rho \left(p_B^\rho(t, \cdot, y_0), \frac{\phi_R(\cdot)}{p_B^{\rho, \epsilon}(t, \cdot, y_0)} \right).$$

Proof: From Lemma 1.3.3 of [11] and Lemma 6.1, we see that $p_B^\rho(t, x, y_0)$ as a function of $x \in B$ is in \mathcal{F}_ρ^B . By Lemma 1.3.4 of [11], we have

$$\begin{aligned} & -\mathcal{E}^\rho\left(p_B^\rho(t, \cdot, y_0), \frac{\phi_R(\cdot)}{p_B^{\rho, \epsilon}(t, \cdot, y_0)}\right) \\ &= \lim_{h \downarrow 0} \frac{1}{h} \left(p_B^\rho(t+h, \cdot, y_0) - p_B^\rho(t, \cdot, y_0), \frac{\phi_R(\cdot)}{p_B^{\rho, \epsilon}(t, \cdot, y_0)} \right) \\ &= \lim_{h \downarrow 0} \frac{1}{h} \left(p_B^{\rho, \epsilon}(t+h, \cdot, y_0) - p_B^{\rho, \epsilon}(t, \cdot, y_0), \frac{\phi_R(\cdot)}{p_B^{\rho, \epsilon}(t, \cdot, y_0)} \right) \\ &= \lim_{h \downarrow 0} \frac{1}{h} \sum \left(\frac{p_B^{\rho, \epsilon}(t+h, x, y_0)}{p_B^{\rho, \epsilon}(t, x, y_0)} - 1 \right) \phi_R(x) \mu_x^\rho. \end{aligned}$$

Moreover,

$$G'(t) = \lim_{h \rightarrow 0} \frac{1}{h} \sum \left(\log p_B^{\rho, \epsilon}(t+h, x, y_0) - \log p_B^{\rho, \epsilon}(t, x, y_0) \right) \phi_R(x) \mu_x^\rho.$$

Let

$$F(h) = \left[\log p_B^{\rho, \epsilon}(t+h, x, y_0) - \log p_B^{\rho, \epsilon}(t, x, y_0) - \left(\frac{p_B^{\rho, \epsilon}(t+h, x, y_0)}{p_B^{\rho, \epsilon}(t, x, y_0)} - 1 \right) \right] \phi_R(x) \mu_x^\rho.$$

Then

$$F'(h) = \frac{\partial p_B^{\rho, \epsilon}(t, x, y_0)}{\partial t} \left(p_B^{\rho, \epsilon}(t, x, y_0) - p_B^{\rho, \epsilon}(t+h, x, y_0) \right) \frac{\phi_R(x)}{p_B^{\rho, \epsilon}(t+h, x, y_0) p_B^{\rho, \epsilon}(t, x, y_0)}.$$

Now the lemma follows easily from using the mean value theorem, Lemma 6.1 and the dominated convergence theorem. \square

Proof of Proposition 2.10: Recall that $R \in [2d, 4d]$. With the help of the above results and Proposition 2.9, Proposition 2.10 follows from similar arguments as in Proposition 4.9 of [1], Theorem 3.4 of [8], or Theorem 2.5 of [10]. \square

7 Appendix Two

In this appendix, we prove Proposition 2.9. If B is an open cube in \mathcal{S} , we define \overline{B} to be the union of all closed cubes in \mathbb{R}^d with centers in B and equal side length ρ^{-1} , and \tilde{B} to be the interior of \overline{B} . If f is defined on \mathcal{S} , we define \tilde{f} as the extension of f to \mathbb{R}^d :

$$\tilde{f}(x) = f([x]_\rho),$$

where $[x]_\rho = (\rho^{-1}[\rho x^1], \dots, \rho^{-1}[\rho x^d])$ for $x = (x^1, \dots, x^d) \in \mathbb{R}^d$. Similarly, we can define $\tilde{C}^\rho(\cdot, \cdot)$ as the extension of $C^\rho(\cdot, \cdot)$ to $\mathbb{R}^d \times \mathbb{R}^d$.

With the above notation, the Poincaré inequality in Lemma 2.3 can be written as follows.

Lemma 7.1 *For any open cube B in \mathcal{S} with side length $2r$, there is a constant c independent of ρ and B such that*

$$\int_{\tilde{B}} (\tilde{f}(x) - \tilde{f}_{\tilde{B}})^2 dx \leq c r^\alpha \int_{\widehat{\kappa_5 B}} \int_{\widehat{\kappa_5 B}} (\tilde{f}(y) - \tilde{f}(x))^2 \tilde{C}^\rho(x, y) \rho^{2d} dx dy,$$

where $\tilde{f}_{\tilde{B}} = |\tilde{B}|^{-1} \int_{\tilde{B}} \tilde{f}(z) dz$ and $|\tilde{B}|$ is the Lebesgue measure of \tilde{B} .

Then we have the following result.

Lemma 7.2 *For any open cube B in \mathbb{R}^d with side length $2r$, there is a constant c independent of ρ and B such that*

$$\int_B (\tilde{f}(x) - \tilde{f}_B)^2 dx \leq c r^\alpha \int_{2B} \int_{2B} (\tilde{f}(y) - \tilde{f}(x))^2 \tilde{C}^\rho(x, y) \rho^{2d} dx dy.$$

Proof: Lemma 7.1 implies that there are constants c_1 and $k > 1$ such that

$$\int_B (\tilde{f}(x) - \tilde{f}_B)^2 dx \leq c_1 r^\alpha \int_{kB} \int_{kB} (\tilde{f}(y) - \tilde{f}(x))^2 \tilde{C}^\rho(x, y) \rho^{2d} dx dy.$$

Our result then follows from the Jerison's technique in [9] or a well-known argument mentioned in §5.3.1 of [14]. \square

Using similar arguments as in the proof of Theorem 5.3.4 in [14], we obtain the following weighted Poincaré inequality.

Proposition 7.3 *For any open cube B in \mathbb{R}^d with center in \mathcal{S} and side length $2R$, there is a constant c independent of R and ρ such that*

$$\int_B (\tilde{f}(x) - \bar{\tilde{f}})^2 \phi_R(x) dx \leq c R^\alpha \int_B \int_B (\tilde{f}(x) - \tilde{f}(y))^2 \phi_R(x) \wedge \phi_R(y) \tilde{C}^\rho(x, y) \rho^{2d} dx dy,$$

where $\bar{\tilde{f}} = \int_B \tilde{f}(x) \phi_R(x) dx$.

Proof of Proposition 2.9: We see that Proposition 2.9 is trivial when $R \leq 1/\rho$ since both sides of the inequality equal zero. For all R with $R - [R] \in [\frac{1}{4\rho}, \frac{1}{2\rho}]$, by Proposition 7.3, we obtain that

$$\begin{aligned} \sum_B (f(x) - \bar{f})^2 \phi_R(x) \rho^{-d} &\leq \sum_B (f(x) - \bar{\tilde{f}})^2 \phi_R(x) \rho^{-d} \\ &\leq 2^d \int_B (\tilde{f}(x) - \bar{\tilde{f}})^2 \phi_R(x) dx \\ &\leq c_1 R^\alpha \int_B \int_B (\tilde{f}(x) - \tilde{f}(y))^2 \phi_R(x) \wedge \phi_R(y) \tilde{C}^\rho(x, y) \rho^{2d} dx dy \\ &\leq c_2 R^\alpha \sum_B \sum_B (f(x) - f(y))^2 \phi_R(x) \wedge \phi_R(y) C^\rho(x, y), \end{aligned}$$

where the sets B in the second and third inequalities are open cubes in \mathbb{R}^d instead of \mathcal{S} . This implies that there exists a constant c_3 independent of ρ and R such that

$$\sum_B (f(x) - \bar{f})^2 \phi_R(x) \rho^{-d} \leq c_3 R^\alpha \sum_B \sum_B (f(x) - f(y))^2 \phi_R(x) \wedge \phi_R(y) C^\rho(x, y)$$

for all $R \in \bigcup_{n=0}^{\infty} [\frac{n}{\rho} + \frac{1}{4\rho}, \frac{n}{\rho} + \frac{1}{2\rho}]$. It is easy to see that the above inequality also holds when

$$R \in \bigcup_{n=0}^{\infty} [\frac{n}{\rho} + \frac{1}{2\rho}, \frac{n}{\rho} + \frac{1}{\rho}]. \quad \square$$

Acknowledgements

The author would like to thank Professor Richard Bass for suggesting this problem. The author also thanks anonymous referees and the associate editor for helpful comments.

References

- [1] M. T. Barlow, R. F. Bass, Z.-Q. Chen and M. Kassmann, Non-local Dirichlet forms and symmetric jump processes, *Trans. Amer. Math. Soc.*, **361** (2009), 1963–1999.
- [2] R.F. Bass and Z.-Q. Chen, Regularity of harmonic functions for a class of singular stable-like processes, *Math. Z.*, **266** (2010), 489–503.
- [3] R.F. Bass and T. Kumagai, Symmetric Markov chains on \mathbb{Z}^d with unbounded range, *Trans. Amer. Math. Soc.*, **360** (2008), 2041–2075.
- [4] R.F. Bass, T. Kumagai, and T. Uemura, Convergence of Symmetric Markov chains on \mathbb{Z}^d , *Prob. Th. rel. Fields*, **148** (2010), 107–140.
- [5] R.F. Bass and D.A. Levin, Transition probabilities for symmetric jump processes, *Trans. Amer. Math. Soc.*, **354** (2002), 2933–2953.
- [6] E.A. Carlen, S. Kusuoka and D.W. Stroock, Upper Bounds for symmetric markov transition functions, *Ann. Inst. H. Poincaré Probab. Statist.*, **23(2, suppl.)** (1987), 245–287.
- [7] Z.-Q. Chen and T. Kumagai, Heat kernel estimates for stable-like processes on d -sets, *Stoch. Proc. Applic.*, **108** (2003), 27–62.
- [8] Z.-Q. Chen, P. Kim and T. Kumagai, Weighted Poincaré inequality and heat kernel estimates for finite range jump processes, *Math. Ann.*, **342** (2008), 833–883.
- [9] D. Jerison, The Poincaré inequality for vector fields satisfying Hörmander’s condition, *Duke Math. J.*, **53(2)** (1986), 503–523.
- [10] M. Foondun, Heat kernel estimates and Harnack inequalities for some Dirichlet forms with non-local part, *Elect. J. Probab.*, **14** (2009), 314–340.
- [11] M. Fukushima, Y. Oshida and M. Takeda, *Dirichlet forms and symmetric Markov processes*, Volume 19 of de Gruyter Studies in Mathematics, Walter de Gruyter & Co., Berlin, 1994.
- [12] R. Hussein and M. Kassmann, Markov Chain Approximations for Symmetric Jump Processes, *Potential Anal.*, **27** (2007), 353–380.
- [13] J. Moser, On Harnack’s theorem for elliptic differential equations, *Comm. Pure Appl. Math.*, **14** (1961), 577–591.
- [14] L. Saloff-Coste, *Aspects of Sobolev-Type Inequalities*, Cambridge University Press, 2002.
- [15] D.W. Stroock and W. Zheng, Markov chain approximations to symmetric diffusions, *Ann. Inst. Henri. Poincaré-Probab. Statist.*, **33** (1997), 619–649.
- [16] F. Xu, A class of singular symmetric Markov processes, *Potential Anal.*, to appear.